

# HYPERBOLIC DIMENSION OF JULIA SETS OF MEROMORPHIC MAPS WITH LOGARITHMIC TRACTS

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**ABSTRACT.** We prove that for meromorphic maps with logarithmic tracts (e.g. entire or meromorphic maps with a finite number of poles from class  $\mathcal{B}$ ), the Julia set contains a compact invariant hyperbolic Cantor set of Hausdorff dimension greater than 1. Hence, the hyperbolic dimension of the Julia set is greater than 1.

Following [2], we say that an unbounded domain  $T \in \mathbb{C}$  with piecewise smooth boundary and unbounded complement is a logarithmic tract for a continuous function  $f : \overline{T} \rightarrow \mathbb{C}$ , if there exists  $R_0 > 0$ , such that  $|f(z)| > R_0$  for every  $z \in T$ ,  $|f(z)| = R_0$  for every  $z \in \partial T$  and  $f$  on  $T$  is a holomorphic universal covering of  $\{z : |z| > R_0\}$ . Note that every meromorphic map with logarithmic singularity over infinity has a logarithmic tract. In particular, this holds for transcendental meromorphic maps with a finite number of poles (in particular transcendental entire maps) from class  $\mathcal{B}$ . Recall that the class  $\mathcal{B}$  consists of maps  $f$ , for which the set of singularities of  $f^{-1}$  is bounded.

The hyperbolic dimension of the Julia set  $J(f)$  is defined as the supremum of the Hausdorff dimensions (denoted  $\dim_H$ ) of all conformal expanding Cantor repellers contained in  $J(f)$ . Recall that in this setting a conformal expanding Cantor repeller is a compact invariant Cantor set  $X \subset J(f)$ , such that  $(f^k)'|_X > 1$  for some  $k > 0$ . Obviously, the hyperbolic dimension of  $J(f)$  is not greater than its Hausdorff dimension. However, it can be strictly smaller, see [9].

In this note we prove the following.

**Theorem.** *The hyperbolic dimension of the Julia set of a meromorphic map with a logarithmic tract is greater than 1. In particular, the Hausdorff dimension of the set of points with bounded orbits in the Julia set is greater than 1.*

It is known that the Julia set of any entire transcendental map contains non-degenerate continua (see [1]) and hence  $\dim_H(J(f)) \geq 1$ . On the other hand, it can be arbitrarily close to 1 (see [7]). In [8], Stallard proved a remarkable result stating that the Hausdorff dimension of the Julia set of entire maps from the class  $\mathcal{B}$  is greater than 1. The result was extended by Rippon and Stallard in [6] to meromorphic maps with finitely many poles, and by Bergweiler, Rippon and Stallard in [2] to maps with logarithmic tracts.

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The proof of our theorem uses thermodynamic formalism. Note that the result was proved in [9] for the specific family of exponential maps.

Let  $f$  be a meromorphic map with a logarithmic tract  $T$ . It is convenient to look at  $f$  in the logarithmic coordinates, introduced in [4], which are defined as follows. Changing coordinates by translation, we can assume that  $0 \notin \overline{T}$ . Moreover, enlarging  $R_0$  in the definition of the logarithmic tract  $T$ , we can assume that  $\partial T$  is homeomorphic to a straight line, so that  $\overline{T}$  is simply connected and  $\partial T \cup \{\infty\}$  is a Jordan curve in the Riemann sphere. Since  $f$  on  $T$  is a universal cover of  $\{z : |z| > R_0\}$ , we can lift  $f$  by the branches of logarithm to a map

$$F : \bigcup_{s \in \mathbb{Z}} \overline{L_s} \rightarrow \overline{H},$$

where  $L_s = L_0 + 2\pi is$  are unbounded simply connected domains in  $\mathbb{C}$  (lifted tracts), such that  $\partial L_s \cup \{\infty\}$  is a Jordan curve and  $H = \{z : \operatorname{Re}(z) > \ln R_0\}$ . We have

$$\exp \circ F = f \circ \exp,$$

in particular,  $F$  is periodic with period  $2\pi i$ . Note that  $F$  is conformal on each  $L_s$  and maps  $\overline{L_s}$  homeomorphically onto  $\overline{H}$ . Let  $F_s^{-1}$  be the inverse branch of  $F$  mapping  $\overline{H}$  onto  $\overline{L_s}$ . Note that  $F_s^{-1} = F_0^{-1} + 2\pi is$ .

**Lemma 1.** *Let  $\varepsilon > 0$ . Then there exist arbitrarily large  $x > 0$ , such that for every  $s \in \mathbb{Z}$ ,*

$$|(F_s^{-1})'(x)| > \frac{1}{x^{1+\varepsilon}}.$$

*Proof.* Let  $x > 0$ . Since  $F_s^{-1}(x) \rightarrow \infty$  as  $x \rightarrow +\infty$ , the curve  $F_s^{-1}([\ln R_0, +\infty))$  has infinite length. Hence, the integral

$$\int_{\ln R_0}^{+\infty} |(F_s^{-1})'(x)| dx$$

is diverging near infinity, which easily shows the lemma.  $\square$

Using the Koebe one-quarter theorem, we obtain the result showed by Eremenko and Lyubich as Lemma 1 in [4].

**Lemma 2.** *For every  $z \in \bigcup_{s \in \mathbb{Z}} L_s$ , we have  $|F'(z)| > \frac{1}{4\pi}(\operatorname{Re}(F(z)) - \ln R_0)$ .*  $\square$

**Corollary 3.** *There exist  $a, b > 0$ , such that for every  $x \geq \ln R_0 + 1$  and  $s \in \mathbb{Z}$ ,*

$$\operatorname{Re}(F_s^{-1}(x)) < a \ln x + b.$$

*Proof.* By Lemma 2,

$$|F_s^{-1}(x) - F_s^{-1}(1 + \ln R_0)| < 4\pi \int_{1+\ln R_0}^x \frac{dt}{t - \ln R_0} = 4\pi \ln(x - \ln R_0).$$

Hence,  $\operatorname{Re}(F_s^{-1}(x)) < 4\pi \ln(x - \ln R_0) + c_0$ , where  $c_0 = \operatorname{Re}(F_s^{-1}(1 + \ln R_0))$  is independent of  $s$ , which implies the assertion.  $\square$

We need one more simple observation:

**Lemma 4.** *If  $z \in \overline{H}$  and  $z \rightarrow \infty$ , then  $\operatorname{Re}(F_s^{-1}(z)) \rightarrow \infty$ .*

*Proof.* It is sufficient to notice that for  $M > 0$  the set  $\overline{L_s} \cap \{z : \operatorname{Re}(z) \leq M\}$  is compact.  $\square$

To prove our theorem, we first construct a suitable Cantor repeller in the logarithmic coordinates.

**Proposition 5.** *There exists a compact set  $J_B \subset \bigcup_{s \in \mathbb{Z}} L_s$  with  $\dim_H(J_B) > 1$ , such that  $F(J_B) \subset \bigcup_{s \in \mathbb{Z}} L_s$  and  $F^2(J_B) = J_B$ . In particular, the forward trajectory of every point in  $J_B$  is well defined and bounded. Moreover,  $(F^n)'(z) \rightarrow \infty$  as  $n \rightarrow \infty$  for every  $z \in J_B$ .*

To prove this proposition, note first that by Lemma 1, there exists an arbitrarily large  $R > 0$ , such that

$$(1) \quad |(F_s^{-1})'(R)| > 1/R^{1+\varepsilon}$$

for a small fixed  $\varepsilon > 0$ . Recall that all preimages

$$v_s = F_s^{-1}(R)$$

are located on the same vertical line

$$\ell = \{v : \operatorname{Re}(v) = r\}$$

and, by Corollary 3, we have

$$r = \operatorname{Re}(v_s) < a \ln R + b \ll R$$

for large  $R$ . On the other hand, given  $D > 0$ , and using Lemma 4, we can require that

$$(2) \quad r = \operatorname{Re}(v_s) > \ln R_0 + 2D,$$

so, in particular, the line  $\ell$  is contained in  $H$ .

Now let us consider all components of  $F^{-1}(\ell)$ , i.e. the sets

$$\ell_u = F_u^{-1}(\ell)$$

for  $u \in \mathbb{Z}$ . Obviously,  $\ell_u \subset L_u$  and again, using Corollary 3, we conclude that

$$(3) \quad \inf_{v \in \ell_u} \operatorname{Re}(v) < a \ln r + b \ll r \ll R,$$

if  $R$  is large. Denote

$$v_{u,s} = F_u^{-1}(v_s) = F_u^{-1} \circ F_s^{-1}(R).$$

and consider the square

$$Q = \left[ \frac{R}{2}, \frac{3R}{2} \right] \times \left[ -\frac{R}{2}, \frac{R}{2} \right]$$

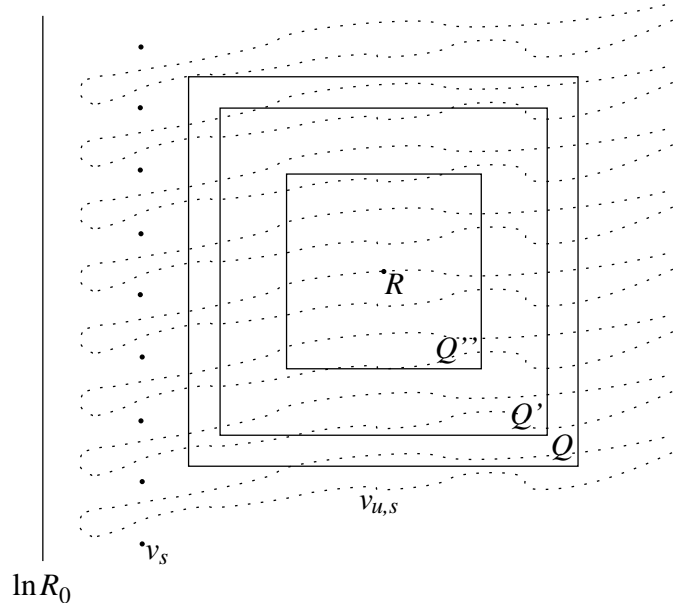
(so that  $R$  is the center of  $Q$  and the sides of  $Q$  have length  $R$ ). See Figure 1.

Next, let us consider the image of  $Q$  under  $F_s^{-1}$ . By Lemma 2,

$$(4) \quad |(F_s^{-1})'(R)| < \frac{4\pi}{R - \ln R_0},$$

and by the Koebe Distortion Theorem (see e.g. [3]), the distortion of  $F_s^{-1}$  on  $Q$  is universally bounded. We can thus write for every  $z \in Q$ ,

$$|(F_s^{-1})'(z)| < C \frac{4\pi}{R - \ln R_0},$$

FIGURE 1. The squares  $Q, Q', Q''$ .

where  $C > 1$  is the distortion constant. Since  $\text{diam}(Q) = \sqrt{2}R$ , we have

$$\text{diam}(F_s^{-1}(Q)) \leq \text{diam}(Q) \frac{4C\pi}{R - \ln R_0} = \frac{4\sqrt{2}\pi CR}{R - \ln R_0} < 5\sqrt{2}\pi C$$

for large  $R$ . Hence, by (2) applied for  $D = 5\sqrt{2}\pi C$ , we get  $F_s^{-1}(Q) \subset H$ , which implies that the inverse branches  $F_u^{-1}$  are defined in  $F_s^{-1}(Q)$ . Moreover, by Lemma 2 and (2),

$$|(F_u^{-1})'(v_s)| < \frac{4\pi}{r - \ln R_0} < \frac{2\pi}{D} < 1,$$

so using (4), we have the following estimate:

$$|(F_u^{-1} \circ F_s^{-1})'(R)| < \frac{4\pi}{R - \ln R_0},$$

which implies, again by the Koebe Distortion Theorem, that

$$|(F_u^{-1} \circ F_s^{-1})'(z)| < \frac{4\pi C}{R - \ln R_0}$$

for all  $z \in Q$ . We conclude that, denoting

$$Q_{u,s} = F_u^{-1} \circ F_s^{-1}(Q),$$

we have

$$(5) \quad \text{diam}(Q_{u,s}) < \text{diam}(Q) \frac{4\pi C}{R - \ln R_0} = \frac{4\sqrt{2}\pi CR}{R - \ln R_0} < D.$$

Now we consider the set  $G$  of indices  $(u, s)$  defined as follows:

$$G = \{(u, s) \in \mathbb{Z}^2 : Q_{u,s} \subset Q\}.$$

We shall prove

**Lemma 6.** *There exists a constant  $C_1 > 0$ , such that for every  $z \in Q$  we have*

$$(6) \quad \sum_{(u,s) \in G} |(F_u^{-1} \circ F_s^{-1})'(z)| > C_1 R^{1-\varepsilon}$$

*Proof.* Let  $Q'' \subset Q' \subset Q$  be the following squares centered at  $R$ :

$$Q' = \left[ \frac{R}{2} + D, \frac{3R}{2} - D \right] \times \left[ -\frac{R}{2} + D, \frac{R}{2} - D \right], \quad Q'' = \left[ \frac{3R}{4}, \frac{5R}{4} \right] \times \left[ -\frac{R}{4}, \frac{R}{4} \right].$$

See Figure 1.

Let  $\mathcal{L}$  be the family of all curves  $\ell_u$  intersecting  $Q''$ . For every  $\ell_u \in \mathcal{L}$ , we denote by  $\ell_{u,Q'}$  a connected component of  $\ell_u \cap Q'$ , such that  $\ell_{u,Q'} \cap Q'' \neq \emptyset$ . Note that every  $\ell_{u,Q'}$  intersects the boundaries of  $Q'$  and  $Q''$ , so

$$(7) \quad \text{length}(\ell_{u,Q'}) \geq \frac{R}{4} - D.$$

Now, consider the subset  $G_u \subset G$  consisting of all  $s \in \mathbb{Z}$ , such that  $v_{u,s} \in Q'$ . Note that by (5),

$$\{(u, s) : s \in G_u\} \subset G.$$

Moreover,  $\ell_{u,Q'} \subset F_u^{-1}(\bigcup_{s \in G_u} B(v_s, 2\pi))$  and the distortion of  $F_u^{-1}$  is uniformly bounded on every ball  $B(v_s, 2\pi)$ . Hence,

$$(8) \quad \text{length}(\ell_{u,Q'}) < C_2 \sum_{s \in G_u} |(F_u^{-1})'(v_s)|$$

for some constant  $C_2 > 0$ . Using (1), (7), (8) and the fact that the distortion of  $F_u^{-1} \circ F_s^{-1}$  is universally bounded on  $Q$ , we get

$$\begin{aligned} \sum_{s \in G_u} |(F_u^{-1} \circ F_s^{-1})'(z)| &> \frac{1}{C} \sum_{s \in G_u} |(F_u^{-1} \circ F_s^{-1})'(R)| \\ &> \frac{1}{CR^{1+\varepsilon}} \sum_{s \in G_u} |(F_u^{-1})'(v_s)| > \frac{1}{CC_2} \frac{R/4 - D}{R^{1+\varepsilon}} > \frac{C_3}{R^\varepsilon}, \end{aligned}$$

for some constant  $C_3 > 0$ , if  $R$  is large. Moreover, using (3), Lemma 4 and the fact  $\ell_u = \ell_0 + 2\pi i$ , we see that the family  $\mathcal{L}$  contains at least  $R/(4\pi)$  curves, so finally we get

$$\sum_{(u,s) \in G} |(F_u^{-1} \circ F_s^{-1})'(z)| \geq \sum_{\{u: \ell_u \in \mathcal{L}\}} \sum_{s \in G_u} |(F_u^{-1} \circ F_s^{-1})'(z)| > \frac{C_3}{4\pi} R^{1-\varepsilon}.$$

□

Now, we are in a position to prove Proposition 5. Let us consider the collection of open, simply connected sets  $\text{int } Q_{u,s}$ , where  $(s, t) \in G$ . It follows from our assumptions that their closures are pairwise disjoint and, by (5) and the definition of  $G$ ,  $Q_{u,s} \subset Q$ . In this way we get a conformal Iterated Function System, formed by strictly contracting, conformal maps  $F_u^{-1} \circ F_s^{-1} : Q \rightarrow Q$ . Such a system has a unique compact invariant set  $J_B$ . Since the  $Q_{u,s}$  are pairwise disjoint,  $J_B$  is a Cantor set. By construction,  $F(J_B) \subset \bigcup_{s \in \mathbb{Z}} L_s$ ,  $F^2(J_B) = J_B$  and  $(F^n)'(z) \rightarrow \infty$  for every  $z \in J_B$ . It is well known that the Hausdorff dimension of  $J_B$  is determined as the unique  $t \in \mathbb{R}$ , such that  $P(t) = 0$ , where  $P(t)$  is so-called pressure function (see e.g. [5]). Let us recall the definition of the pressure function in our setting:

$$P(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{g^n} \|(g^n)'\|^t.$$

Here, we sum up over all possible compositions  $g^n = g_{i_1} \circ \dots \circ g_{i_n}$ , where  $g_{i_j} = F_u^{-1} \circ F_s^{-1}$  for some  $(u, s) \in G$ .

Since the system is expanding, the function  $t \mapsto P(t)$  is strictly decreasing and in order to prove  $\dim_H(J_B) > 1$  it is enough to show that  $P(1) > 0$ . Using Lemma 6 we can estimate the pressure  $P(1)$  as follows:

$$\begin{aligned} P(1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{g^n} \|(g^n)'\| \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \inf_{z \in Q} \sum_{(u,s) \in G} |(F_u^{-1} \circ F_s^{-1})'(z)| \right)^n \geq \log(C_1 R^{1-\varepsilon}) > 0 \end{aligned}$$

for  $\varepsilon < 1$  and  $R$  sufficiently large. Therefore,  $\dim_H(J_B) > 1$ . This ends the proof of Proposition 5.

Proposition 5 immediately implies the following theorem, which proves our main result.

**Theorem 7.** *There exists a conformal expanding Cantor repeller  $X \subset J(f)$  with the Hausdorff dimension greater than 1.*

*Proof.* Let  $X = \exp(J_B \cup F(J_B))$ . Since  $\exp \circ F = f \circ \exp$  and  $J_B \cup F(J_B)$  is  $F$ -invariant,  $X$  is  $f$ -invariant. Moreover, since  $\exp$  is a smooth covering,  $\dim_H(X) = \dim_H(J_B) > 1$ . Finally, since  $\lim_{n \rightarrow \infty} (F^n)'(z) = \infty$  for  $z \in J_B$ , we conclude that also  $\lim_{n \rightarrow \infty} (f^n)'(z) = \infty$  for  $z \in X$ . On the other hand, the trajectory  $f^n(z)$  is bounded. Therefore, the family of iterates  $\{f^n\}$  cannot be normal in any neighbourhood of  $z \in X$ . Consequently,  $X \subset J(f)$ .  $\square$

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